

# Pair dispersion in synthetic fully developed turbulence

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(February 5, 2008)

The Lagrangian statistics of relative dispersion in fully developed turbulence is numerically investigated. A scaling range spanning many decades is achieved by generating a synthetic velocity field with prescribed Eulerian statistical features. When the velocity field obeys Kolmogorov similarity, the Lagrangian statistics is self similar too, and in agreement with Richardson’s predictions. For an intermittent velocity field the scaling laws for the Lagrangian statistics are found to depend on Eulerian intermittency in agreement with a multifractal description. As a consequence of the Kolmogorov law the Richardson law for the variance of pair separation is not affected by intermittency corrections. A new analysis method, based on fixed scale averages instead of usual fixed time statistics, is shown to give much wider scaling range and should be preferred for the analysis of experimental data.

## I. INTRODUCTION

Understanding the statistics of particle pairs dispersion in turbulent velocity fields is of great interest for both theoretical and practical implications. At variance with single particle dispersion, which depends mainly on large scale, energy containing eddies, pair dispersion is driven (at least at intermediate times) by velocity fluctuations at scales comparable with the pair separation. These small scale fluctuations are thought to be independent on the particular large scale flow [1]. Since fully developed turbulence displays well known, non-trivial universal features in the Eulerian statistics of velocity differences [2,3], pair dispersion represents a starting point for the investigation of the general problem of the relationship between Eulerian and Lagrangian properties.

Since the pioneering work by Richardson [4], many efforts have been done to confirm experimentally [2] or numerically [5–7] his law. Nevertheless, the main obstacle to a deep investigation of relative dispersion in turbulence remains the lack of sufficient statistics due to technical difficulties in laboratory experiments and to the moderate inertial range reached in direct numerical simulations.

Moreover, also from an applicative point of view, a deep comprehension of the relative dispersion mechanisms is of fundamental importance for a correct modelization of small scale diffusion and mixing properties.

In this Paper we present a detailed investigation of the statistics of relative dispersion, obtained by numerical simulations of the advection of particle pairs in a synthetic turbulent velocity field with prescribed Eulerian statistical features.

In first place, we deal with the probability distribution of Lagrangian quantities in a self similar, Kolmogorov-like scaling flow where we confirm the Richardson-Obukhov predictions. We then investigate the effects of Eulerian intermittency on Lagrangian statistics where we find “Lagrangian intermittency”, i.e. deviations of the scaling exponents from the Richardson values. These effects cannot be captured by dimensional arguments alone. The simplest step beyond dimensional considerations is to extend the multifractal description, which is successfully used for Eulerian statistics, to Lagrangian quantities. We will see that our numerical simulations confirm the Lagrangian multifractal predictions.

The intermittency corrections to relative dispersion are however rather small. Moreover, they can be hidden by the finite scaling range. Huge Reynolds numbers are necessary in order to discriminate clearly the scaling exponents. Within this framework, we propose a new methodology for the analysis of relative dispersion data, which deals with Lagrangian statistics at fixed spatial separation between particles. In particular, the statistics of “doubling times” – the time that two particles spend while doubling their separation – seems a very promising tool in data analysis.

In Section II we address the problem of relative dispersion in fully developed, homogeneous and isotropic turbulence. In Section III the construction of a self similar synthetic turbulent field is described. In Section IV results on the Lagrangian statistics of particle pairs advected by a non-intermittent velocity field are presented. In Section V the

method of doubling times is introduced and its advantages with respect to the usual fixed time statistics are discussed. In Section VI is shown how to build an intermittent velocity field, and the effects of the Eulerian field on Lagrangian statistics are discussed. In Section VII conclusions are drawn.

## II. THE RICHARDSON LAW

We consider the dispersion of a pair of particles passively advected by an homogeneous, isotropic, fully developed turbulent field. Due to the incompressibility of the velocity field the particles will, on average, separate from each other [8,9]. The statistics of pairs separation is conveniently summarized by the probability density function of distances between couples of particles at a given time,  $p(\mathbf{R}, t)$ , called *distance neighbor function* by Richardson [4].

In view of the diffusive effect exerted by the turbulent motion on the advected particles, Richardson argued that the time evolution of the distance neighbor function could be described by a proper diffusion equation

$$\frac{\partial p(\mathbf{R}, t)}{\partial t} = \frac{\partial}{\partial R_j} \left( K(R) \frac{\partial p(\mathbf{R}, t)}{\partial R_j} \right) \quad (1)$$

with a  $R$  dependent scalar turbulent diffusivity  $K(R)$ . From a collection of experimental data, Richardson was able to obtain his celebrated “4/3” law:

$$K(R) = \alpha R^{4/3} \quad (2)$$

where  $\alpha$  is a constant. This choice for the diffusivity relied mainly on empirical grounds (dependence of the vertical eddy diffusivity in the atmosphere with the altitude). The fortunate Richardson choice of a rational exponent demonstrates his faith in an underlying universal physical mechanism of simple form.

It is easy to realize that in three dimensions the solution of equation (1) is

$$p(\mathbf{R}, t) = N(\alpha t)^{-9/2} \exp \left( -\frac{9R^{2/3}}{4\alpha t} \right) \quad (3)$$

where  $N$  is a normalization factor, which immediately leads to the growth laws for the average separation

$$\langle R^{2n}(t) \rangle = \int d\mathbf{R} R^{2n} p(\mathbf{R}, t) \sim t^{3n}. \quad (4)$$

A discussion of the validity of diffusive approximations to the problem of pair dispersion in turbulence can be found in chapter 24.4 of Monin-Yaglom [2].

The scaling (4) can actually be derived by a simple dimensional argument due to Obukhov [2] starting from Kolmogorov similarity law for velocity increments in fully developed turbulence

$$\langle |\delta \mathbf{v}^{(E)}(\mathbf{R})| \rangle = \langle |\mathbf{v}(\mathbf{x} + \mathbf{R}) - \mathbf{v}(\mathbf{x})| \rangle \sim R^{1/3} \quad (5)$$

with  $R = |\mathbf{R}|$ . The particle pair separates according to

$$\frac{d\mathbf{R}}{dt} = \delta \mathbf{v}^{(L)}(\mathbf{R}) \quad (6)$$

where  $\delta \mathbf{v}^{(L)}$  represents the velocity difference evaluated along the Lagrangian trajectories. Assuming  $\delta v^{(L)}(R) \simeq |\delta \mathbf{v}^{(E)}(R)|$  from (5) one obtains  $dR^2/dt \sim R \delta v_{\parallel}^{(L)}(R) \sim R^{4/3}$  and hence the Richardson law (4). The assumption that the Lagrangian velocity difference has the same (Kolmogorov) scaling as the Eulerian one relies on the intuitive idea that the main contribution to the rate of separation comes from eddies of a size comparable to that of the separation itself.

After having outlined the classical arguments on relative dispersion, two remarks are in order. First, that the determination of specific functional dependencies – such as the shape of the distance-neighbor-function  $p(\mathbf{R}, t)$  – lies beyond the possibilities of similarity hypotheses, and thus calls for additional hypotheses, like equation (1). Second, diffusive approximations have no possibility of capturing intermittency effects on Lagrangian statistics. We have thus to face with supplementary hypotheses which have a considerable degree of arbitrariness, and whose content can be mainly judged a posteriori. In this respect the use of synthetic velocity fields represents a flexible framework for the study of the detailed features of the statistics of pair dispersion.

### III. THE SYNTHETIC TURBULENT FIELD

The generation of a synthetic turbulent field which reproduces the relevant statistical features of fully developed turbulence is not an easy task. Indeed to obtain a physically sensible evolution for the velocity field one has to take into account the fact that each eddy is subject to the action of all other eddies. Actually the overall effect amounts only to two main contributions, namely the sweeping exerted by larger eddies and the shearing due to eddies of comparable size. This is indeed a substantial simplification, but nevertheless the problem of properly mimicking the effect of sweeping is still unsolved.

It is relatively easy to construct a spatial, time independent, self affine or multiaffine velocity field in any dimension. In order to let the particle separate one has then to introduce some time dependence (at least in two dimensions). This has been done in previous synthetic simulations either by adding a large scale flow which sweeps the particles on a quenched background of eddies [6], or by moving arbitrarily the Eulerian structures [7]. Of course, both the solutions are rather unphysical.

To get rid of these difficulties we shall limit ourselves to the generation of a synthetic velocity field in *Quasi-Lagrangian* (QL) coordinates [10], thus moving to a frame of reference attached to a particle of fluid  $\mathbf{r}_1(t)$ . This choice bypasses the problem of sweeping, since it allows to work only with relative velocities, unaffected by advection. As a matter of fact there is a price to pay for the considerable advantage gained by discarding advection, and it is that only the problem of two-particle dispersion can be well managed within this framework. The properties of single-particle Lagrangian statistics cannot, on the contrary, be consistently treated.

The QL velocity differences are defined as

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{u}(\mathbf{r}_1(t) + \mathbf{r}, t) - \mathbf{u}(\mathbf{r}_1(t), t) , \quad (7)$$

where the reference particle moves according to

$$\dot{\mathbf{r}}_1(t) = \mathbf{u}(\mathbf{r}_1(t), t) . \quad (8)$$

These velocity differences have the useful property that their single-time statistics are the same as the Eulerian ones whenever considering statistically stationary flows [10]. For fully developed turbulent flows, in the inertial interval of length scales where both viscosity and forcing are negligible, the QL longitudinal velocity differences show the scaling behavior

$$\left\langle \left| \mathbf{v}(\mathbf{r}) \cdot \frac{\mathbf{r}}{r} \right|^p \right\rangle \sim r^{\zeta_p} \quad (9)$$

where the exponent  $\zeta_p$  is a convex function of  $p$ , and  $\zeta_3 = 1$ . This scaling behavior is a distinctive statistical property of fully developed turbulent flows that we shall reproduce by means of a synthetic velocity field.

In the QL reference frame the first particle is at rest in the origin and the second particle is at  $\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{R}$ , advected with respect to the reference particle by the relative velocity

$$\mathbf{v}(\mathbf{R}, t) = \mathbf{u}(\mathbf{r}_1(t) + \mathbf{R}, t) - \mathbf{u}(\mathbf{r}_1(t), t) \quad (10)$$

By this change of coordinates the problem of pair dispersion in an Eulerian velocity field has been reduced to the problem of single particle dispersion in the velocity difference field  $\mathbf{v}(\mathbf{r}, t)$ . This yields a substantial simplification: it is indeed sufficient to build a velocity difference field with proper scaling features in the radial direction only, that is along the line that joins the reference particle  $\mathbf{r}_1(t)$  – at rest in the origin of the QL coordinates – to the second particle  $\mathbf{r}_2(t) = \mathbf{r}_1(t) + \mathbf{R}(t)$ . To appreciate this simplification, it must be noted that actually all moments of velocity differences  $\mathbf{u}(\mathbf{r}_1(t) + \mathbf{r}', t) - \mathbf{u}(\mathbf{r}_1(t) + \mathbf{r}, t) = \mathbf{v}(\mathbf{r}', t) - \mathbf{v}(\mathbf{r}, t)$  should display power law scaling in  $|\mathbf{r}' - \mathbf{r}|$ . Actually these latter differences never appear in the dynamics of pair separation, and so we can limit ourselves to fulfill the weaker request (9). Needless to say, already for three particle dispersion one needs a field with proper scaling in all directions.

We limit ourselves to the two-dimensional case, where we can introduce a stream function for the QL velocity differences

$$\mathbf{v}(\mathbf{r}, t) = \nabla \times \psi(\mathbf{r}, t) . \quad (11)$$

The extension to a three dimensional velocity field is not difficult but more expensive in terms of numerical resources. Under isotropic conditions, the stream function can be decomposed in radial octaves as

$$\psi(\mathbf{r}, \theta, t) = \sum_{i=1}^N \sum_{j=1}^n \frac{\phi_{i,j}(t)}{k_i} F(k_i r) G_{i,j}(\theta). \quad (12)$$

where  $k_i = 2^i$ . Following a heuristic argument, one expects that at a given  $r$  the stream function is essentially dominated by the contribution from the  $i$  term such that  $r \sim 2^{-i}$ . This locality of contributions suggests a simple choice for the functional dependencies of the “basis functions”:

$$F(x) = x^2(1-x) \quad \text{for } 0 \leq x \leq 1 \quad (13)$$

and zero otherwise,

$$G_{i,1}(\theta) = 1, \quad G_{i,2}(\theta) = \cos(2\theta + \varphi_i) \quad (14)$$

and  $G_{i,j} = 0$  for  $j > 2$  ( $\varphi_i$  is a quenched random phase). It is worth remarking that this choice is rather general because it can be derived from the lowest order expansion for small  $r$  of a generic streamfunction in Quasi-Lagrangian coordinates.

It is easy to show that, under the usual locality conditions for IR convergence,  $\zeta_p < p$  [11], the leading contribution to the  $p$ -th order longitudinal structure function  $\langle |v_r(r)|^p \rangle$  stems from  $M$ -th term in the sum (12),  $\langle |v_r(r)|^p \rangle \sim \langle |\phi_{M,2}|^p \rangle$  with  $r \simeq 2^{-M}$ . If the  $\phi_{i,j}(t)$  are stochastic processes with characteristic times  $\tau_i = 2^{-2i/3} \tau_0$ , zero mean and  $\langle |\phi_{i,j}|^p \rangle \sim k_i^{-\zeta_p}$ , the scaling (9) will be accomplished. An efficient way of to generate  $\phi_{i,j}$  is [12]:

$$\phi_{i,j}(t) = g_{i,j}(t) z_{1,j}(t) z_{2,j}(t) \cdots z_{i,j}(t) \quad (15)$$

where the  $z_{k,j}$  are independent, positive definite, identically distributed random processes with characteristic time  $\tau_k$ , while the  $g_{i,j}$  are independent stochastic processes with zero mean,  $\langle g_{i,j}^2 \rangle \sim k_i^{-2/3}$  and characteristic time  $\tau_i$ . The scaling exponents  $\zeta_p$  are determined by the probability distribution of  $z_{i,j}$  via

$$\zeta_p = \frac{p}{3} - \log_2 \langle z^p \rangle. \quad (16)$$

As a last remark we note that by simply fixing the  $z_{i,j} = 1$  we recover the Kolmogorov scaling.

In this Paper we shall consider either synthetic turbulent fields without corrections to Kolmogorov scaling, i.e.  $\zeta_p = p/3$ , either fields whose intermittency corrections to the Kolmogorov scaling, i.e., nonlinear  $\zeta_p$ , are close to the experimental three dimensional turbulence exponents [13], see Table I.

#### IV. LAGRANGIAN STATISTICS IN ABSENCE OF EULERIAN INTERMITTENCY

When the advecting velocity field has Kolmogorov scaling, one expects Richardson law (4) to hold for the variance of pair separation. This is indeed very well verified in our numerical simulations, as shown in Figure 1, over a range of separations of approximately 3 decades. Observe that in the present example the effective Reynolds number, defined as  $Re = (k_N/k_1)^{4/3}$  is already rather large,  $Re \simeq 10^{10}$ . The scaling range in the relative dispersion statistics is found to be strongly reduced with respect to that of structure functions.

Furthermore, by similarity arguments, the distance neighbor function  $p(\mathbf{R}, t)$  should assume the self similar, isotropic form (in two-dimensions)

$$p(\mathbf{R}, t) = Ct^{-3} \Phi(R/t^{3/2}) \quad (17)$$

where  $C$  is a normalization factor and  $\Phi(\xi)$  is a universal function whose shape is not predicted by similarity hypotheses. We checked the validity of (17) by rescaling the numerically obtained distance neighbor functions with respect to the theoretical average separation  $\langle R \rangle \sim t^{3/2}$ . The different rescaled pdf's, see Figure 2, collapse onto a unique curve, which represents the shape of the universal function  $\Phi(\xi)$  with  $\xi = R/t^{3/2}$ . The continuous line is the Richardson prediction  $\Phi(\xi) \sim \exp(-b\xi^{2/3})$  rephrasing eq. (3), which is in good agreement with the data. This result supports the empirical picture that the relative diffusion can be regarded as generated by a turbulent diffusivity, according to (1).

Another interesting statistics which can be investigated is the pdf of Lagrangian velocity differences along the trajectory,  $p_L(\delta v|t)$ . Dimensional arguments give  $\langle \delta v^2 \rangle \sim t$  showing the accelerating nature of Richardson dispersion

[2]. In Figure 3 we plot the numerical computed pdf of  $\delta v/t^{1/2}$ . The clear collapse of the curves for different times  $t$  demonstrate the validity of the scaling assumption.

Most of the previous works concerning the validation of the Richardson law have been focused mainly on the numerical prefactor (Richardson constant [2]). We show that our results are realistic also for the Richardson constant  $G_\Delta$  defined from the pair dispersion law  $R^2(t) = G_\Delta \bar{\epsilon} t^3$ . The value of  $\bar{\epsilon}$  can be obtained from the second order Eulerian structure function, which reads  $S_2^{(E)}(R) = \langle |\delta v_\parallel^{(E)}(R)|^2 \rangle = C_L \bar{\epsilon}^{2/3} R^{2/3}$  where  $C_L$  is a universal constant related to the Kolmogorov constant. According to the experimental measurements we fix  $C_L = 2.0$ , leading to  $G_\Delta = 0.190 \pm 0.005$  for the Richardson constant which is in agreement with previous values [2,6].

It is worth remarking that, in the above expressions,  $\bar{\epsilon}$  is *not* the average energy dissipation rate of the turbulent flow. If it was so, we should be forced to conclude that in flows where no energy flux is present, no relative dispersion takes place. Of course this is not the case, and the energy flux does not play a significant role in determining the intensity of the turbulent diffusivity. Strong turbulent diffusivity arises as a consequence of incompressibility and non stationarity of the velocity field. Thus  $\bar{\epsilon}$  has thus to be intended as a dimensional factor which simply rescales all the dimensional expressions with respect to the large scale velocity  $v_0$  and lengthscale  $\ell_0$ .

## V. DOUBLING TIME STATISTICS

A closer look to Figure 1 shows that the power-law scaling regime  $\langle R^2(t) \rangle \sim t^3$  is observed only well inside the inertial range. To explain this effect let us consider a series of pair dispersion experiments, in which a couple of particles is released at a separation  $R_0$  at time  $t = 0$ . At a fixed time  $t_1$ , as customarily is done, we perform an average over all different experiments to compute  $\langle R^2(t_1) \rangle$ . But, unless  $t_1$  is large enough that all particle pairs have “forgotten” their initial conditions, our average will be biased. This is the origin of the flattening of  $\langle R^2(t) \rangle$  for small times, that we can call a crossover from initial condition to self similarity. In an analogous fashion there is a crossover for large times, of the order of the integral time-scale, since some couples might have reached a separation larger than the integral scale, and thus diffuse normally, meanwhile other pairs still lie within the inertial range, biasing the average and, again, flattening the curve  $\langle R^2(t) \rangle$ . This effect is particular evident for lower Reynolds numbers, as shown in Figure 4 for a simulation with  $Re \simeq 10^6$ . This correction to a pure power law is far from being negligible for instance in experimental data where the inertial range is generally limited due to the Reynolds number and the experimental apparatus. For example, references [7,14] show quite clearly the difficulties that may arise in numerical simulations with the standard approach.

To overcome this difficulty we propose an alternative approach which is based on the statistics at fixed scale, instead of at fixed time. The method is in the spirit of a recently introduced generalization of the Lyapunov exponent to finite size perturbation (Finite Size Lyapunov Exponent) which has been successfully applied in the predictability problem [15] and in the diffusion problem [16]. Given a set of thresholds  $R_n = R_0 2^n$  within the inertial range, we compute the “doubling time”  $T(R_n)$  defined as the time it takes for the particle separation to grow from one threshold  $R_n$  to the next one  $R_{n+1}$ . Averages are then performed over many dispersion experiments. The outstanding advantage of averaging at a fixed separation scale is that it removes all crossover effects, since all sampled pairs belong to the inertial range.

The scaling properties of the doubling times is obtained by a simple dimensional argument. The time it takes for particle separation to grow from  $R$  to  $2R$  can be estimate as  $T(R) \sim R/\delta v(R)$ ; we thus expect for the inverse doubling times the scaling

$$\left\langle \frac{1}{T^p(R)} \right\rangle \sim \frac{\langle \delta v(R)^p \rangle}{R^p} \sim R^{-2p/3} \quad (18)$$

In Figure 5 the great enhancement in the scaling range achieved by using “doubling times” is well evident.

The conclusion that can be drawn by this simple example is that the doubling time statistics allows a much better estimation of the scaling exponent with respect to the standard, fixed time, statistics. This property will be used in the following section for investigating the scaling law of the relative dispersion in presence of Eulerian intermittency.

## VI. THE EFFECT OF EULERIAN INTERMITTENCY

In former literature there are very few attempts to investigate possible corrections stemming from Eulerian intermittency [17–20]. This is quite surprising compared with the enormous amount of literature concerning the intermittency

correction for the Eulerian statistics [3,21]. This mismatch is partly due to the difficulty of having experimental checks of proposed theoretical corrections. The use of synthetic velocity fields provides a first benchmark which is extremely easier and less expensive compared to experiments and direct numerical simulations.

In Figure 6 we report the Lagrangian longitudinal structure functions  $S_p^{(L)}(r) = \langle (\delta v_{\parallel}^{(L)}(r))^p \rangle$ , which are computed recording the Lagrangian velocity difference whenever the pair separation equals  $r$ . Observe the wide inertial range over more than 10 decades, corresponding to an integral Reynolds number  $Re \simeq 10^{10}$ . Let us remark that, due to the average growth of particle separation, also the first order Lagrangian structure function is non zero. The most interesting and non-trivial result is that scaling exponents for the Lagrangian structure functions show up to be almost exactly the same  $\zeta_p$  of the Eulerian case (Table I). In terms of the multifractal formalism [3,22], this result is restated by saying that the fractal dimension  $D(h)$  for the Lagrangian velocity statistics is the same of the Eulerian one.

With this preliminary results in mind, we can extend the dimensional argument for the Richardson law to the intermittent case by using the multifractal representation.

From the definition

$$\frac{d}{dt} \langle R^p \rangle = \langle R^{p-1} \delta v_{\parallel}^{(L)} \rangle \quad (19)$$

by using the multifractal representation for the velocity differences, we can write

$$\frac{d}{dt} \langle R^p \rangle \sim \int dh R^{p-1+h+3-D(h)}. \quad (20)$$

The time it takes for the pair separation to reach the scale  $R$  is dominated by the largest time in the process and can be dimensionally estimated as  $t \sim R/\delta v \sim R^{1-h}$ . Averaging over many realizations gives the expression

$$\frac{d}{dt} \langle R^p \rangle \sim \int dh t^{\frac{p+2+h-D(h)}{1-h}} \quad (21)$$

which, once evaluated by steepest descent method, gives the final result  $\langle R^p \rangle \sim t^{\alpha_p}$  with scaling exponents

$$\alpha_p = \inf_h \left[ \frac{p+3-D(h)}{1-h} \right] \quad (22)$$

In the case of intermittent velocity field, the relative dispersion displays non linear scaling exponent  $\alpha_p$  (see Table I). However there is an interesting result, already obtained in [17], for the case  $p = 2$ . From the general multifractal formalism one has that  $3 - D(h) \geq 1 - 3h$  and the equality is satisfied for the scaling exponent  $h_3$  which realizes the third order structure function  $\zeta_3 = 1$ . From (22) follows that  $\alpha_2 = 3$  and thus we have that the Richardson law  $\langle R^2 \rangle \sim t^3$  is not affected by intermittency corrections, while the other moments in general are. We note that the previous argument leading to (22) is just one dimensional reasonable assumption which can be justified only a posteriori by numerical simulations. Other different assumptions are possible [17–19] leading to different predictions.

The scaling exponents satisfy the inequality  $\alpha_p/p < 3/2$  for  $p > 2$ : this amounts to say that, as time goes by, the right tail of the pdf of the separation  $R(t)$  becomes less and less broad. In other words, due to the effect of Eulerian intermittency, particle pairs are more likely to stay close to each other than to experience a large separation.

In Figure 7 we show the result of the computation of  $\langle R^p(t) \rangle$  for different  $p$ . We find that  $\langle R^2(t) \rangle$  displays a clear  $t^3$  scaling law, but the scaling region becomes smaller for higher moments, making the determination of the exponents  $\alpha_p$  rather difficult. To overcome this difficulty we plot the moments  $\langle R^p(t) \rangle$  compensated with  $\langle R^2(t) \rangle^{\alpha_p/3}$  which should result constant according to (22). For comparison we plot also the moment  $p = 4$  compensated assuming normal scaling, i.e.  $\langle R^4(t) \rangle \sim \langle R^2(t) \rangle^2$ . It is evident that prediction (22) is compatible with our numerical data, while the Richardson scaling (4) is not. To be more quantitative, in Table I we report the numerical  $\alpha_p$  directly obtained by a fit of  $\langle R^p(t) \rangle$ . The numerical exponents, although affected by large uncertainty, are rather close to the theoretical ones.

The time doubling analysis discussed in Section V reveals very useful in the case of Eulerian intermittency. To see how the scaling of the doubling times are affected by Eulerian intermittency, we can give a dimensional estimate of the doubling time as  $T(R) \sim R/\delta v(R)$  and thus see that it fluctuates with the velocity fluctuations. After averaging over many realizations we can write

$$\left\langle \frac{1}{T^p(R)} \right\rangle \sim \int dh R^{p(h-1)} R^{3-D(h)} \simeq R^{\zeta_p-p} \quad (23)$$

from which follows that the doubling time statistics contains the same information on the Eulerian intermittency as the relative dispersion exponents (22). Let us remark that also in this case there is the exponent  $\zeta_3 - 3 = -2$  unaffected by Eulerian intermittency.

As reported in Figure 8 our prediction is very well verified in numerical simulations. The plot of the compensated inverse time statistics clearly discriminates between multiaffine scaling (23) and affine scaling (18) (here reported only for  $p = 4$ ). Note that also in the present intermittent simulations, the scaling region for the inverse time statistics is wider than that of Figure 7 and the scaling exponent can be determined with much higher accuracy. In Table I we report the theoretical exponent  $\beta_p = \zeta_p - p$  of (23) compared with the direct numerical fit. The agreement is within 2%.

### A. Fluctuating characteristic times

It must be pointed out that the construction of the synthetic field here proposed shows an inconsistency with the expected statistical properties imposed by Navier-Stokes equations. Indeed, the characteristic time the eddies of size  $R$  are chosen to be  $\tau(R) \sim R^{2/3}$ . Actually, since by dimensional argument  $\tau(R) \sim R/\delta v(R)$ , as long as  $\delta v(R)$  are fluctuating quantities, the characteristic times also fluctuate. This effect is definitely enhanced in presence of intermittency, where the average characteristic times do not show simple scaling [10]. This finding could shed some doubt on the validity of the results obtained up to here.

We thus turn to a popular dynamical model of turbulence which seems to have been developed exactly for our purpose. Shell Models [21] are deterministic models which displays a dynamic energy cascade. The model is built in terms of shell variables  $u_n$  which represent the velocity differences in a wavenumber octave  $k_n = k_0 2^n$  in Quasi-Lagrangian coordinates. With a suitable choice of parameters, the model develops chaotic dynamics which is responsible of intermittency correction to the Kolmogorov scaling exponents remarkably close to the experimental data [21]. Being a complete deterministic system, the Shell Model also displays dynamic eddy turnover times with the same statistics expected for Navier-Stokes turbulence [23].

The particular model we use is a recently proposed Shell Model [24] for the complex variables  $u_n$ :

$$\frac{du_n}{dt} = ik_n \left( u_{n+2} u_{n+1}^* - \frac{1}{4} u_{n+1} u_{n-1}^* + \frac{1}{8} u_{n-1} u_{n-2} \right) - \nu k_n^2 u_n + f_n \quad (24)$$

where  $\nu$  is the viscosity and  $f_n$  is a forcing term restricted to the first two shells.

The QL stream function (12) can be written in terms of the  $u_n$  variables by taking  $\phi_{i,1} = \Re(u_i)$ ,  $\phi_{i,2} = \Im(u_i)$ . The scaling exponents for the Eulerian structure functions  $\zeta_p$  are numerically computed and listed in Table I. In Figure 9 we report the statistics of Lagrangian doubling times compensated with the theoretical scaling (23). Also in this case there is evidence of anomalous scaling in agreement with the multifractal prediction, confirming our previous finding with the stochastic velocity field. This result indicates that the relative dispersion statistics is not very sensible to the details of the time dependence of the Eulerian velocity field. We think this is the main reason for which previous authors [6,7] were able to observe Richardson dispersion even with rather artificial Eulerian dynamics.

## VII. CONCLUSIONS

In this paper we have proposed a simple and efficient method for generating a time dependent, turbulent-like velocity difference field in Quasi-Lagrangian coordinates. The synthetic flow is constructed with prescribed two-point scaling properties (structure functions) thus allowing extensive investigations of the effects of intermittency on Lagrangian pair dispersion.

For non intermittent, Kolmogorov scaling, turbulence we find that the original Richardson approach, based on a diffusion equation for relative separation, in agreement with our simulations. In the case of intermittent turbulence we have found that relative dispersion displays anomalous scaling exponents and cannot be any longer described as a self-similar process. Relative dispersion intermittency can be captured by a natural extension of the multifractal representation to Lagrangian quantities which leads to the theoretical prediction of Lagrangian scaling exponents.

We have suggested a new approach based on the Lagrangian doubling times which extends the scaling range with respect to the standard, fixed time statistics, and is thus very promising for data analysis.

The present work are a first step towards the clarification of Lagrangian-Eulerian relationship in fully developed turbulence. It would be extremely interesting to check our claims by mean of direct numerical simulations or laboratory experiments.

We thank L. Biferale for useful discussions. This work has been partially supported by the INFM (Progetto di Ricerca Avanzata TURBO) and by the MURST (program 9702265437).

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$p$	$\zeta_p$	$\zeta_p^{num}$	$\alpha_p$	$\alpha_p^{num}$	$\zeta_p - p$	$\beta_p^{num}$	$\zeta_p^{SM}$
1	0.390	0.39	1.59	1.56	-0.610	-0.62	0.39
2	0.719	0.74	3.00	2.94	-1.281	-1.28	0.73
3	1.0	1.04	4.32	4.27	-2.0	-1.99	1.01
4	1.245	1.30	5.58	5.58	-2.755	-2.73	1.26
5	1.461	1.54	6.80	6.88	-3.539	-3.49	1.49
6	1.655	1.74	7.99	8.17	-4.345	-4.23	1.71

TABLE I. Theoretical and numerical fitted scaling exponent for the simulations with intermittent velocity field. The number of shells is  $N = 30$  corresponding to an integral Reynolds number  $Re \simeq 10^{10}$ .  $\zeta_p$ : Eulerian structure functions theoretical scaling exponents.  $\zeta_p^{num}$ : Lagrangian structure function numerical scaling exponents.  $\alpha_p$ : theoretical relative dispersion scaling exponents.  $\alpha_p^{num}$ : numerical relative dispersion scaling exponents.  $\zeta_p - p$ : theoretical doubling time scaling exponents.  $\beta_p^{num}$ : numerical doubling time scaling exponents.  $\zeta_p^{SM}$ : Eulerian scaling exponents for the Shell Model.

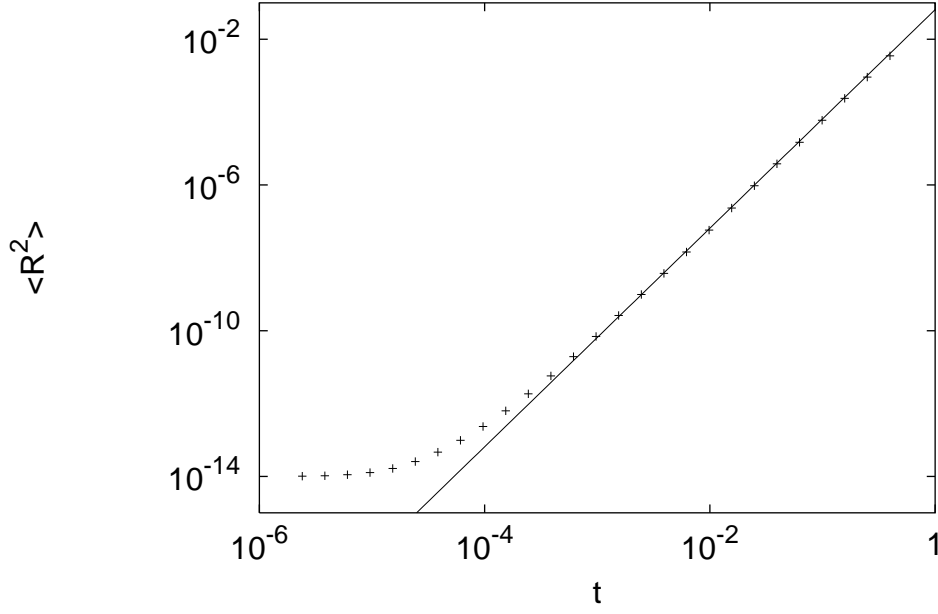


FIG. 1. Average variance of pair separation  $\langle R^2(t) \rangle$  for simulation with  $N = 30$  octaves averaged over  $10^4$  realizations. The continuous line represent the Richardson scaling  $t^3$ .

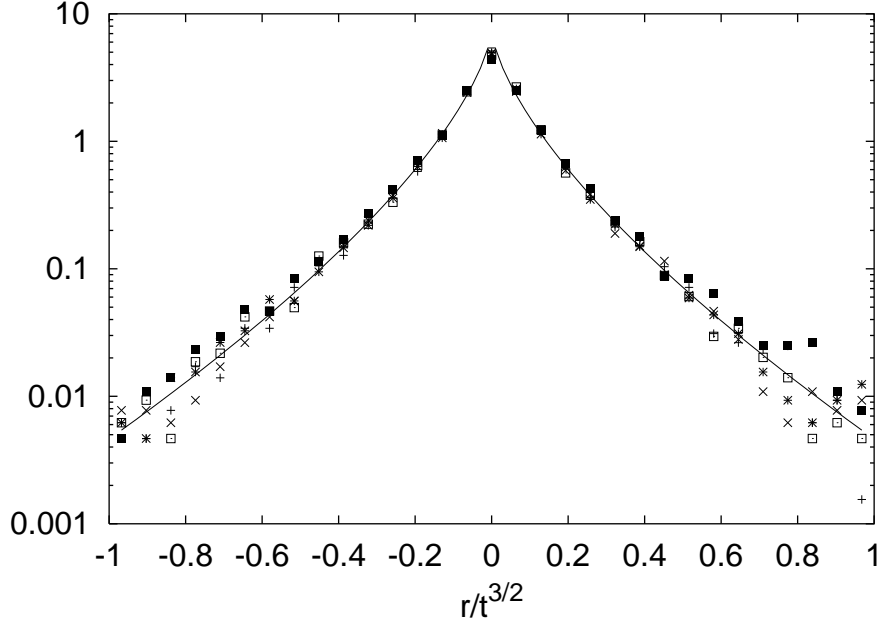


FIG. 2. Probability distribution functions of separations  $R$  (distance neighbor function) rescaled with the theoretical scaling  $t^{3/2}$  for  $10^{-3} < t < 0.25$  (i.e. in the scaling region of previous figure). The continuous line represents the Richardson distribution (3).

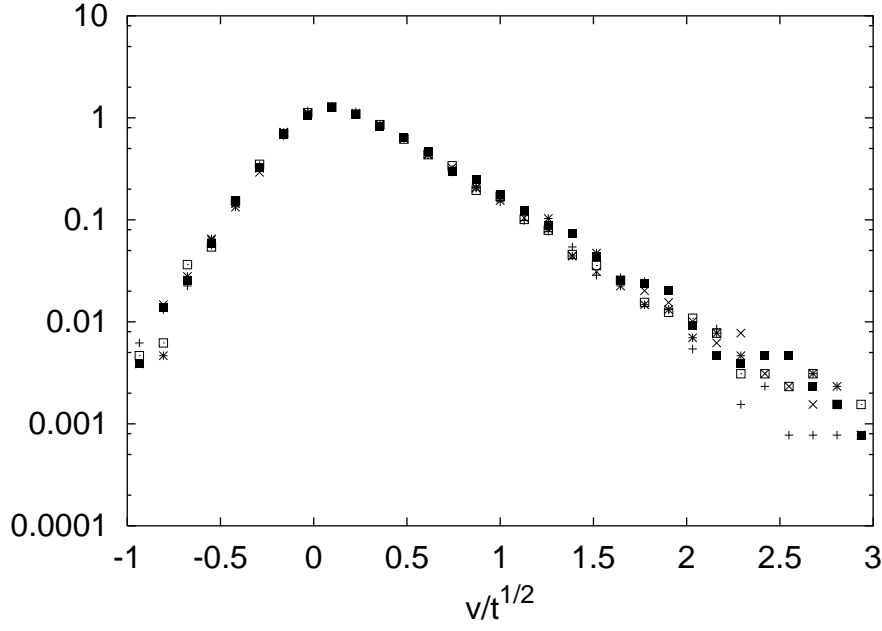


FIG. 3. Probability distribution functions of Lagrangian longitudinal velocity differences at fixed  $t$  rescaled with the theoretical scaling  $t^{1/2}$  for the same values of  $t$  as in figure 2.

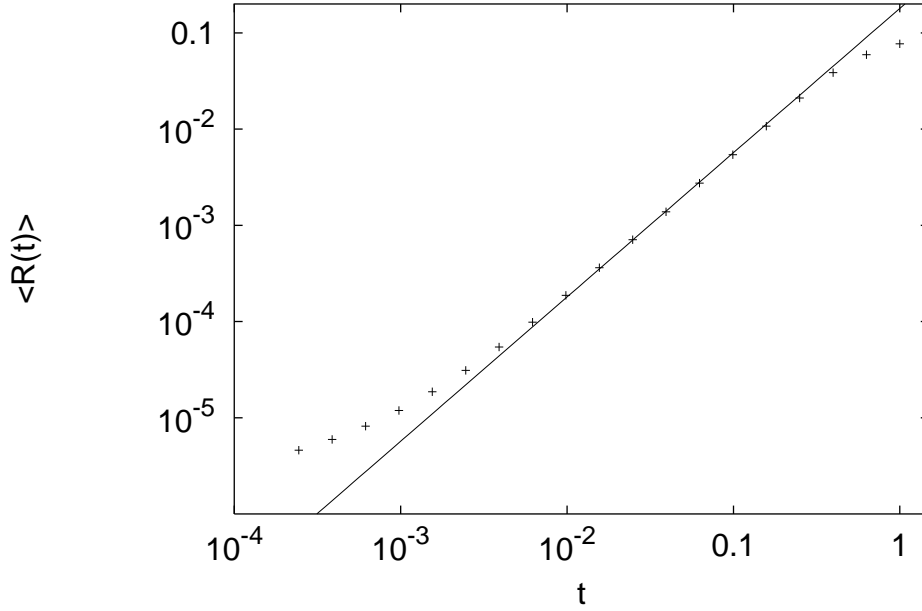


FIG. 4. Relative dispersion  $\langle R(t) \rangle$  for  $N = 20$  octaves simulation averaged over  $10^4$  realizations. The line is the theoretical Richardson scaling  $t^{3/2}$ .

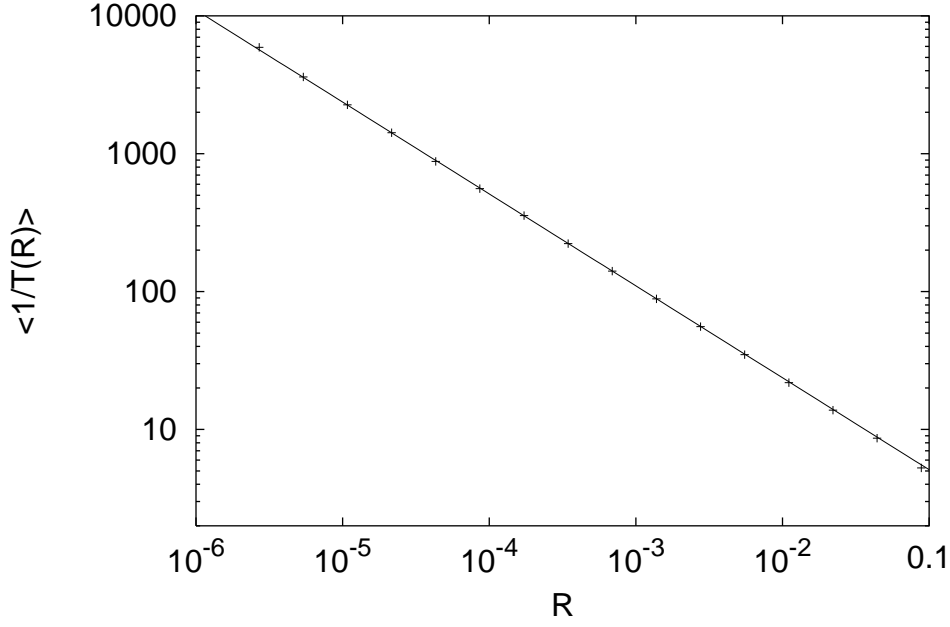


FIG. 5. Average inverse doubling time  $\langle 1/T(R) \rangle$  for the same simulation of Figure 4. Observe the enhanced scaling region. The line is the theoretical Richardson scaling  $R^{-2/3}$  (18).

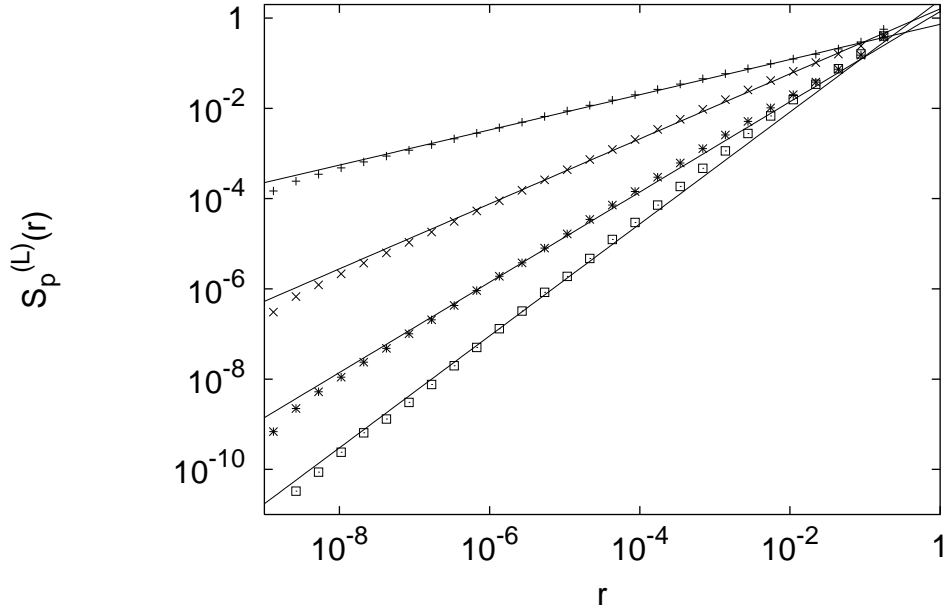


FIG. 6. Longitudinal Lagrangian structure functions  $S_p^{(L)}(R)$  for  $p = 1, 2, 3, 4$  (from top to bottom) for  $N = 30$  octaves intermittent velocity field. Average is over  $10^5$  particle pairs. The continuous lines represents the theoretical scaling with exponents  $\zeta_p$  given in Table I.

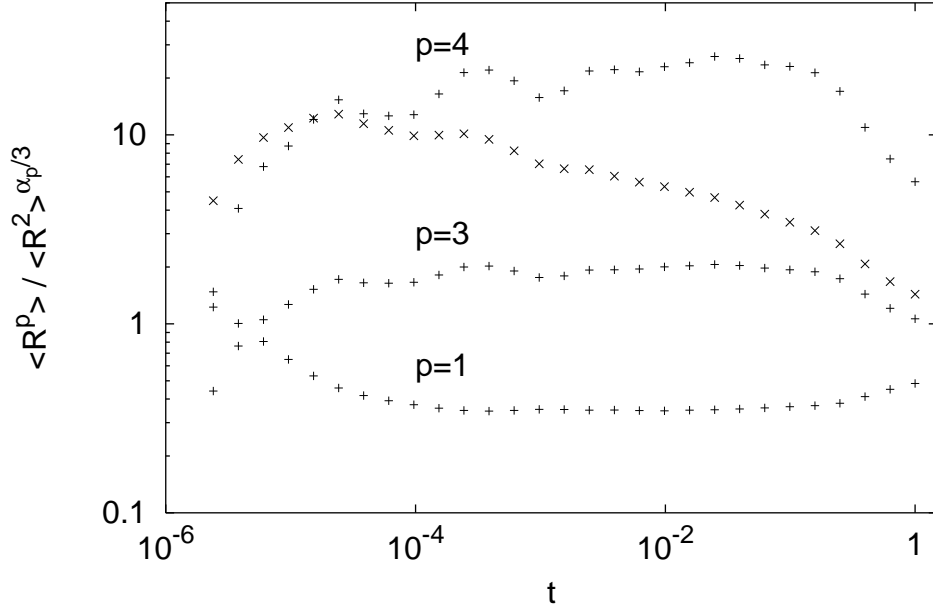


FIG. 7. Relative dispersion  $\langle R^p(t) \rangle$  rescaled with  $\langle R^2(t) \rangle^{\alpha_p/3}$  for  $p = 1, 3, 4$  (+). The almost constant plateau indicates a relative scaling in agreement with prediction (22). For comparison we also plot  $\langle R^4(t) \rangle$  rescaled with the non intermittent prediction  $\langle R^2(t) \rangle^2$  (x) clearly indicating a deviation from normal scaling.

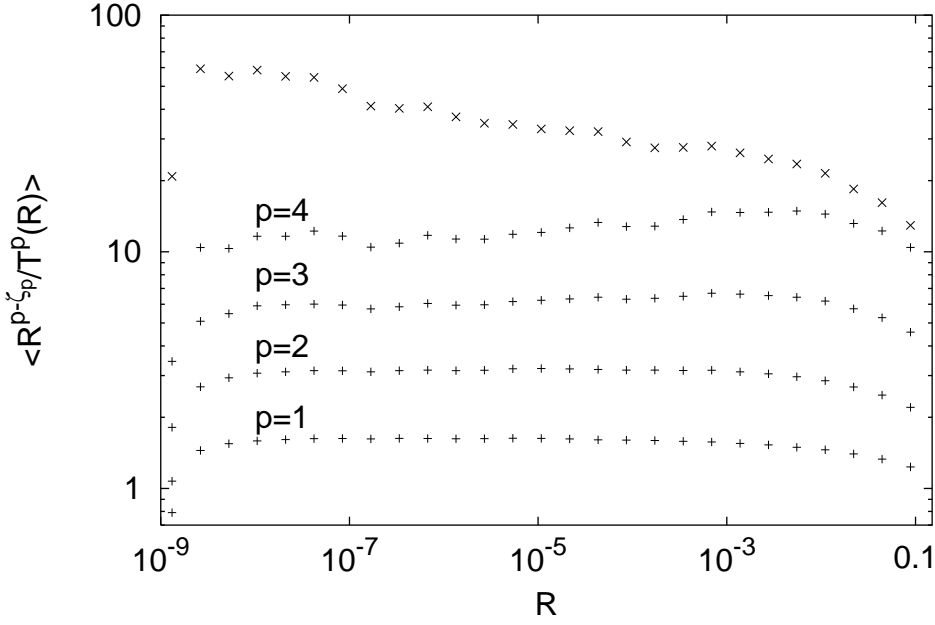


FIG. 8. Inverse doubling times statistics  $\langle 1/T^p(R) \rangle$  compensated with the multifractal prediction (18)  $R^{\zeta_p-p}$  for  $p = 1, 2, 3, 4$  (+). Inverse doubling times  $\langle 1/T^4(R) \rangle$  compensated with the non intermittent prediction  $R^{-8/3}$  (x).

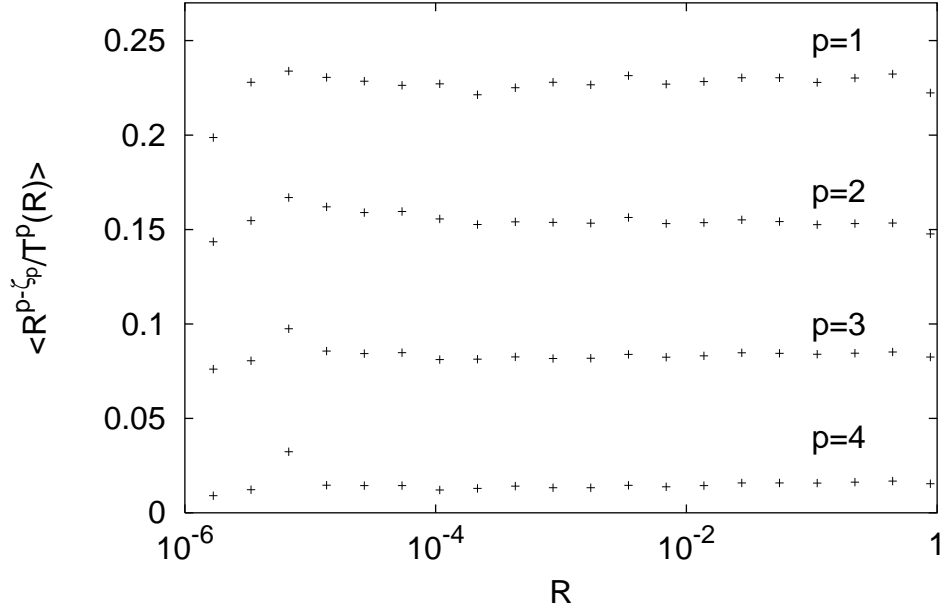


FIG. 9. Inverse time statistics  $\langle 1/T^p(R) \rangle$  compensated with the multifractal prediction  $R^{\zeta_p - p}$  for the Shell Model simulation with  $N = 24$  shells,  $\nu = 10^{-8}$  and  $k_0 = 0.05$ . The average is over  $10^4$  realizations of particle pairs.